The Bivariate Normal Distribution
Case Study: Hedging a Loan Guarantee

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July, 2023

We will construct a buy-and-hold hedge portfolio where we are short one put option (illiquid share) and long another (liquid share). In this white paper we will use the mathematics from the bivariate normal distribution to calculate the expected payoff mean and variance on this hedge portfolio. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are tasked with calculating the expected payoff on a hedge portfolio that is short a put option on ABC Company (a loan guarantee) and long a put option on XYZ Company (a correlated asset or index). We are given the following model parameters...

Table 1: Model Parameters

Symbol	Description	Value
S_0	ABC Company current share price	30.0000
H_0	XYZ Company current share price	20.0000
$X(s)$	ABC Company debt balance per share	25.0000
$X(h)$	XYZ Company put option exercise price	10.0000
$P(s)$	Guarantee value per share at time zero	3.7000
$P(h)$	XYZ put option price per share at time zero	1.9000
$\kappa(s)$	ABC Company - Expected return - mean	0.1133
$\sigma(s)$	ABC Company - Expected return - volatility	0.3000
$\phi(s)$	ABC Company - Expected return - dividend yield	0.0296
$\kappa(h)$	XYZ Company - Expected return - mean	0.1398
$\sigma(h)$	XYZ Company - Expected return - volatility	0.4000
$\phi(h)$	XYZ Company - Expected return - dividend yield	0.0488
α	Risk-free rate	0.0407
ρ	Pairwise return correlation coefficient	0.8000
Γ	Hedge ratio	0.7500
T	Term in years	5.0000

Note: The hedge ratio is the number of XYZ Company put options (long position) used to hedge one ABC Company put option (short position).

Answer the following questions...

Question 1: What is the unhedged portfolio's payoff mean and variance at time T?

Question 2: What is the hedged portfolio's payoff mean and variance at time T?

Question 3: How effective is the hedge?

Question 4: Graph portfolio payoff variance at time T as a function of the hedge ratio in the range $[0,3]$.

Generic Asset Price Equations

We will define the variable κ to the continuous-time asset return mean, the variable ϕ to be the continuous-time dividend yield, the variable σ to be asset return volatility, and the variable T to be time in years. The equations for asset return mean and variance over the time interval $[0,T]$ are...

$$m=\left(\kappa-\phi-\frac{1}{2}\,\sigma^2\right)T \;\text{...and...}\; v=\sigma^2\,T \tag{1}$$

We will define the variable A_T to be random asset price at time T and the variable θ to be random asset return over the time interval $[0,T]$. Using Equation (1) above, the equation for random asset price at time T as a function of asset price at time zero is...

$$A_T=A_0\,\mathrm{Exp}\left\{\theta\right\} \;\text{...where...}\; \theta\sim N\left[m,v\right] \tag{2}$$

Using Equations (1) and (2) above, the equation for the probability density function for normally-distributed random asset returns is...

$$\mathrm{PDF}=\sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\frac{(\theta-m)^2}{v}\right\}\delta\theta \tag{3}$$

We will define the variable P_T to be the payoff on a put option on the asset at time T and the variable X_T to be the exercise price of that put option. Using Equation (2) above, the equation for option payoff at time T is...

$$P_T=\mathrm{Max}\left[X_T-A_T,0\right] \tag{4}$$

To remove the [Max] function from the equation above we need to know the point at which the option is at-the-money. Using Equations (2) and (4) above, the equation for the asset return where the option is at-the-money is...

$$\text{if...}\; A_T=X_T \;\text{...then...}\; \theta=\ln\left(\frac{X_T}{A_0}\right) \tag{5}$$

Using the equations above, the equation for expected option payoff at time T is...

$$\int\limits_{-\infty}^{c}\sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\frac{(\theta-m)^2}{v}\right\}\left(X_T-A_T\right)\delta\theta \;\text{...where...}\; c=\ln\left(\frac{X_T}{A_0}\right) \tag{6}$$

Constructing The Hedge Portfolio

We will define the variable ρ to be the asset return correlation coefficient. The equation for correlation is...

$$\rho=\text{Correlation of the random return on ABC Company and the random return on XYZ Company} \tag{7}$$

Using Equation (1) above and the data in Table 1 above, the equations for return mean and variance for the two securities underlying our two put options are...

$$\begin{aligned} m(s)&=\left(\kappa(s)-\phi(s)-\frac{1}{2}\,\sigma(s)^2\right)T \;\text{...and...}\; v(s)=\sigma(s)^2\,T \\ m(h)&=\left(\kappa(h)-\phi(h)-\frac{1}{2}\,\sigma(h)^2\right)T \;\text{...and...}\; v(h)=\sigma(h)^2\,T \end{aligned} \tag{8}$$

Using Equations (2) and (8) above and the data in Table 1 above, the equations for the prices at option expiration of our two securities underlying our two put options are...

$$\begin{aligned} A(s)_T&=A(s)_0\,\mathrm{Exp}\left\{\theta(s)\right\} \;\text{...where...}\; \theta(s)\sim N\left[m(s),v(s)\right] \\ A(h)_T&=A(h)_0\,\mathrm{Exp}\left\{\theta(h)\right\} \;\text{...where...}\; \theta(h)\sim N\left[m(h),v(h)\right] \end{aligned} \tag{9}$$

We will define the random variable $P(s,h)$ to be the payoff on the hedge portfolio at time T. Using Equation (9) above and the data in Table 1 above, the equation for hedge portfolio payoff is...

$$P(s,h)=-\mathrm{Max}\left[X(s)-A(s)_T,0\right]+\Gamma\,\mathrm{Max}\left[X(h)-A(h)_T,0\right] \tag{10}$$

Hedged Portfolio Payoff Mean and Variance

We defined the variable ρ to be the pairwise correlation of ABC Company returns with XYZ Company returns. The equation for the probability density function of the bivariate normal distribution is... [1]

$$\text{PDF(s,h)} = \frac{1}{2\,\pi}\sqrt{\frac{1}{(1-\rho^2)\,v(s)\,v(h)}}\,\text{Exp}\left\{-\frac{1}{2}\left[\frac{(\theta(s)-m(s))^2}{v(s)}+\frac{\left(\theta(h)-m(h)-\rho\sqrt{\frac{v(h)}{v(s)}}\,(\theta(s)-m(s))\right)^2}{(1-\rho^2)\,v(h)}\right]\right\} \tag{11}$$

Using Equations (10) and (11) above, the equation for the first moment of the distribution of hedge portfolio payoffs is...

$$\text{FM} = \mathbb{E}\Big[P(s,h)\Big] = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\text{PDF(s,h)}\,P(s,h)\,\delta\theta(s)\,\delta\theta(h) \tag{12}$$

Using Equations (10) and (11) above, the equation for the second moment of the distribution of hedge portfolio payoffs is...

$$\text{SM} = \mathbb{E}\Big[P(s,h)^2\Big] = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\text{PDF(s,h)}\,P(s,h)^2\,\delta\theta(s)\,\delta\theta(h) \tag{13}$$

Using Equations (12) and (13) above, the mean and variance of the distribution of hedge portfolio payoffs is...

$$\text{HP Payoff Mean} = \text{FM} \;\; \text{...and...} \;\; \text{HP Payoff Variance} = \text{SM} - [\text{FM}]^2 \tag{14}$$

The Answers To Our Hypothetical Problem

Using Equation (1) above and the data in Table 1 above, the equations for return mean and variance are...

$$\begin{aligned} m(s) &= \left(0.1133 - 0.0296 - \frac{1}{2}\times 0.30^2\right)\times 5 = 0.1938 \;\; \text{...and...} \;\; v(s) = .30^2\times 5 = 0.4500 \\ m(h) &= \left(0.1398 - 0.0488 - \frac{1}{2}\times 0.40^2\right)\times 5 = 0.0549 \;\; \text{...and...} \;\; v(s) = .40^2\times 5 = 0.8000 \end{aligned} \tag{15}$$

Question 1: What is the unhedged portfolio's payoff mean and variance at time T?

The answer to the question is... [3]

$$\text{Unhedged mean} = -2.21 \;\; \text{...and...} \;\; \text{Unhedged variance} = 19.26 \tag{16}$$

Question 2: What is the hedged portfolio's payoff mean and variance at time T?

The moments of the distribution of hedge portfolio payoffs are... (See VBA code below)

$$\text{First moment} = -1.35 \;\; \text{...and...} \;\; \text{Second moment} = 11.89 \tag{17}$$

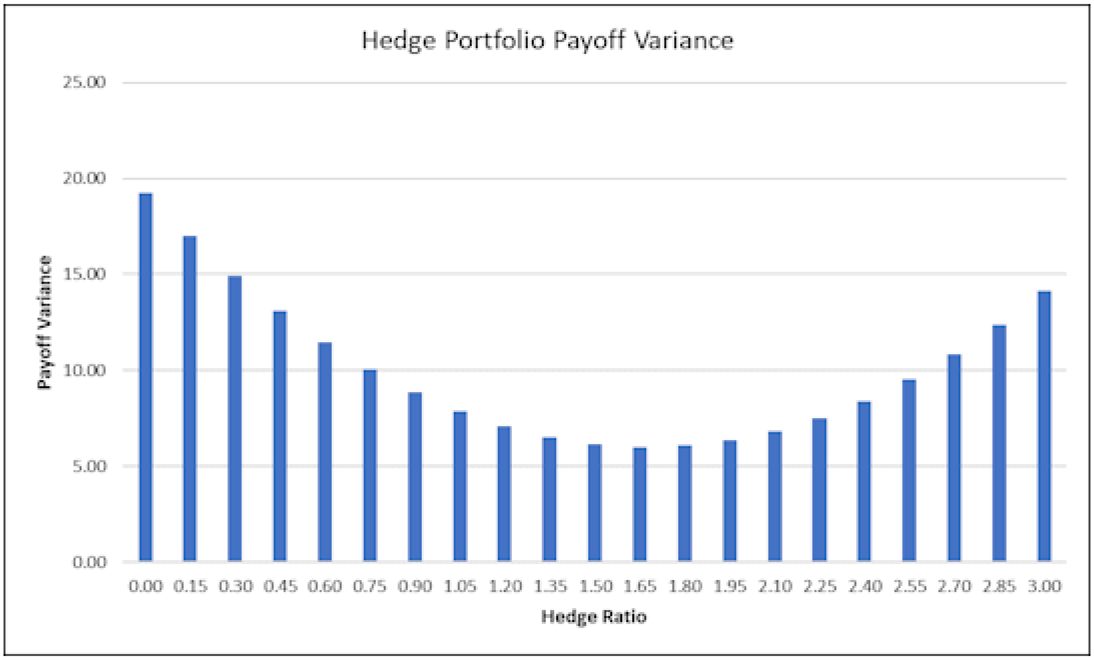
The answer to the question is...

$$\text{Hedged mean} = \text{FM} = -1.35 \;\; \text{...and...} \;\; \text{Hedged variance} = \text{SM} - [\text{FM}]^2 = 10.08 \tag{18}$$

Question 3: How effective is the hedge?

The portfolio payoff mean increased from 2.21 to -1.35 (an increase of 39%) and the portfolio payoff variance decreased from 19.26 to 11.89 (a decrease of 38%). The hedge was effective in reducing portfolio variance.

Question 4: Graph portfolio payoff variance at time T as a function of the hedge ratio in the range $[0, 3]$.



References

[1] Gary Schurman, *The Bivariate Normal Distribution - No Correlation,* July, 2023.

[2] Gary Schurman, *Univariate Ordinary Least Squares Estimator,* May, 2011.

[3] Gary Schurman, *The Mathematics Of Stock Option Valuation - Option Payoff Distribution,* July, 2023.

Listing 1: VBA Code

```
'Name: CBVNormalPDF
'Purpose: Returns the bivariate-normal distribution probability density.
Private Function CBVNormalPDF(a As Double, b As Double, mean_a As Double, mean_b As Double, _
variance_a As Double, variance_b As Double, rho As Double) As Double
    'Declare calculation variables.
    Dim mValue As Double
    Dim mValue01 As Double
    Dim mValue02 As Double
    Dim mValue03 As Double
    Dim mValue04 As Double
    Dim mValue05 As Double
    Dim mValue06 As Double
    'Define calculation variable values.
    mValue01 = 1 / (2 * PIValue)
    mValue02 = (1 - rho ^ 2) * variance_a * variance_b
    mValue03 = (a - mean_a) ^ 2
    mValue04 = (b - mean_b - rho * Sqr(variance_b / variance_a) * (a - mean_a)) ^ 2
    mValue05 = variance_a
    mValue06 = (1 - rho ^ 2) * variance_b
    mValue = mValue01 * Sqr(1 / mValue02)
    mValue = mValue * Exp(-1 / 2 * (mValue03 / mValue05 + mValue04 / mValue06))
    'Return value to caller.
    CBVNormalPDF = mValue
End Function

'Name: BVNMoments
'Purpose: Returns first and second moment from our hedge portfolio.
Public Function BVNMoments(price_a As Double, price_b As Double, exprice_a As Double, _
exprice_b As Double, mean_a As Double, mean_b As Double, variance_a As Double, _
variance_b As Double, gamma As Double, rho As Double) As Variant
    'Declare calculation variables.
    Dim mAPrice As Double
    Dim mBPrice As Double
    Dim mALowerBound As Double
    Dim mAUpperBound As Double
    Dim mBLowerBound As Double
    Dim mBUpperBound As Double
    Dim mAPayoff As Double
    Dim mBPayoff As Double
    Dim mPayoff As Double
    Dim mPDF As Double
    Dim mStepSize As Double
    Dim mNumberSteps As Integer
    Dim a As Double, da As Double
    Dim b As Double, db As Double
    Dim FM As Double
    Dim SM As Double
    'Declare calculation variables.
    Dim moments As Variant
    ReDim moments(1 To 2, 1 To 1)
    'Define integral parameters.
    mNumberSteps = 100
    mAUpperBound = mean_a + Sqr(variance_a) * 4
    mALowerBound = mean_a - Sqr(variance_a) * 4
    mBUpperBound = mean_b + Sqr(variance_b) * 4
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    mBLowerBound = mean_b - Sqr(variance_b) * 4
    da = (mAUpperBound - mALowerBound) / (mNumberSteps - 1)
    db = (mBUpperBound - mBLowerBound) / (mNumberSteps - 1)
    'Calculate distribution moments.
    For a = mAUpperBound To mALowerBound Step -da
        For b = mBUpperBound To mBLowerBound Step -db
            mPDF = CBVNormalPDF(a, b, mean_a, mean_b, variance_a, variance_b, rho)
            mAPrice = price_a * Exp(a)
            mBPrice = price_b * Exp(b)
            mAPayoff = MAX(exprice_a - mAPrice, 0)
            mBPayoff = gamma * MAX(exprice_b - mBPrice, 0)
            mPayoff = -mAPayoff + mBPayoff
            FM = FM + mPayoff * mPDF * da * db
            SM = SM + mPayoff ^ 2 * mPDF * da * db
        Next b
    Next a
    'Return array to caller.
    moments(1, 1) = FM
    moments(2, 1) = SM
    BVNMoments = moments
End Function

'Name: CNDF
'Purpose: Return cumulative normal distribution function.
Private Function CNDF(z As Double, mean As Double, variance As Double) As Double
    CNDF = Application.WorksheetFunction.NormDist(z, mean, Sqr(variance), True)
End Function

'Name: PIValue
'Purpose: Returns the value of pi.
Private Function PIValue() As Double
    PIValue = Application.WorksheetFunction.Pi()
End Function
```

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